Int. J. Heat Mass Transfer. Vol. 20, pp. 1195–1204. Pergamon Press 1977. Printed in Great Britain

APPLICATION OF THE FINITE ELEMENT METHOD TO NATURAL CONVECTION HEAT TRANSFER FROM THE OPEN VERTICAL CHANNEL

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(*Received* 12 *November* 1975 *and in revised form* 18 *February* 1977)

Abstract—The finite element method is used together with the variational principle of the local potential to obtain approximate numerical solutions for steady-state laminar natural convection from the open vertical channel with uniform wall temperature. The natural boundary conditions generated by the variational technique are invoked at the channel entrance and exit. The restriction has so far been observed that fluid velocity may be vertical only. Temperatures and upward velocities respectively are obtained for Rayleigh and Grashof number ranges from 0.1 to 10^7. Nusselt and Reynolds number correlations compare very closely with those for the finite difference solution by Bodoia and Osterle, and less well with a solution for the single vertical wall.

NOMENCLATURE

A^e, area of element e;
b, channel width;
B, width-to-height ratio of channel $= \dfrac{b}{l}$;
E, functional defined by $\phi = \dfrac{\partial E}{\partial \tau}$;
E_s, part of functional pertaining to the system boundary;
E_{mom}, and E_{energy}, equivalent functionals for momentum and energy equations respectively;
g, gravitational acceleration;
h, heat absorbed by fluid from entrance to exit;
H, dimensionless heat-transfer rate;
k, thermal conductivity of fluid;
l, height of channel;
N, proportional error in pressure fall;
p, fluid pressure;
p_∞, hydrostatic fluid pressure;
Q, dimensionless volume flow rate;
t, time;
τ, dimensionless time $= \dfrac{t\nu Gr}{b^2}$;
T, temperature;
T_w, T_∞, channel wall, ambient fluid temperature;
θ, dimensionless temperature $= \dfrac{T - T_\infty}{T_w - T_\infty}$;
θ^0, dimensionless temperature which minimises the functional E;
θ_i, dimensionless temperature for node i;
u, v, fluid velocity in x, y, directions;
U, dimensionless fluid velocity in x-direction $= \dfrac{ub^2}{\nu Grl}$;
U^0, dimensionless velocity which minimises the functional E;
U_i, dimensionless velocity in x-direction for node i;
x, y, z, cartesian coordinate system;
X, Y, dimensionless cartesian coordinate system $= \dfrac{x}{l}, \dfrac{y}{b}$;
α, convection coefficient of heat transfer $= \dfrac{h}{l(T_w - T_\infty)}$;
β, fluid coefficient of expansion;
κ, thermal diffusivity;
ν, kinematic viscosity;
ρ, density;
ϕ, thermodynamic potential;
Gr, Grashof number;
Nu, Nusselt number;
Pr, Prandtl number $= \dfrac{\nu}{\kappa}$;
Ra, Rayleigh number $= Gr \times Pr$;
Re, Reynolds number;
$[N]$, shape function;
$[K]$, matrix K;
$\{\theta\}$, column matrix θ.

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1. INTRODUCTION

THE FINITE element method is, like the finite difference method, a discretisation technique which approximates a problem described by a system of differential equations, by a large number of algebraic equations which may be solved by digital computer. Perhaps the most common way to apply the finite element method is to take a physical problem which may be stated in the form of an extremum or variational principle. An early textbook in this field is that by Zienkiewicz [1] in 1967. An instructive treatment of variational methods is that by Schechter [2] in 1967. Glansdorff and Prigogine [3] in 1964 developed a variational principle—the concept of the local potential—which deals with transport phenomena. Currently the status of the local potential is somewhat in question. However MacDonald [4] has surveyed several papers applying this principle to boundary layer problems, all of which employed the boundary-layer idealisation that x and y may be replaced by $\eta = y/\delta(x)$. MacDonald presented conditions for equivalence of the procedures in these papers with a simpler solution method, which conditions are satisfied in all the cases reviewed, and suggested the quality of results is comparable with that obtained by Karman–Pohlhausen type integral techniques. The above boundary-layer idealisation of course is not used in the present work.

The term "channel" denotes the fluid-filled region between two inward-facing parallel plane solid surfaces, or "walls". The walls normally have the same shape and are exactly opposed. The channel is open all around to fluid flow. In the present case, the walls are vertical. With the origin of co-ordinates at the foot of one wall, the x-direction is assigned vertical, and the y-direction normal to the walls, since the principal flow will be vertical. The channel has considerable "depth" in the z-direction, so that the system is two-dimensional. The only thermal condition considered is that of uniform steady temperature T_w of both walls, and a uniform undisturbed fluid temperature T_∞, which is lower than T_w. All conditions are symmetrical about the channel midplane and steady with time.

It has been well established by means of similarity transformations, the integral method, or experimental studies that for the elementary problem of narrow channels, at one extreme, and the single vertical plane wall at the other extreme, the heat-transfer relations may be put in the respective forms:

$$Nu = A \times (Gr \times Pr) \tag{1}$$

$$Nu = B \times (Gr \times Pr)^{1/4} \tag{2}$$

where A is constant, and B is constant for a given value of Pr. We define non-dimensional parameters for the vertical channel as follows:

$$\left.\begin{aligned} Ra &= \frac{g\beta(T_w - T_\infty)b^4}{\nu\kappa l} \\ Gr &= \frac{g\beta(T_w - T_\infty)b^4}{\nu^2 l} \\ Nu &= \frac{\alpha b}{k} \\ Re &= \frac{\bar{u}b^2}{\nu l}. \end{aligned}\right\} \tag{3}$$

Results for the narrow channel from Elenbaas [5], Bodoia and Osterle [6] and Dyer and Fowler [7] give in terms of the groups in equations (3)

$$Nu = \tfrac{1}{24}Ra \tag{4}$$

$$Re = \tfrac{1}{12}Gr. \tag{5}$$

A similarity solution for the vertical plane wall by Ostrach [8] gives for $Pr = 0.70$ the heat-transfer correlation

$$Nu = 0.510 Ra^{1/4}. \tag{6}$$

From Ostrach's data, also for $Pr = 0.70$, a corresponding fluid flow correlation has recently been obtained by Al Rawi [9]:

$$Re = 0.231\, Gr^{1/2}. \tag{7}$$

The mean upward velocity used in forming the Reynolds number is that within the 2% velocity boundary layer at the top of the wall. Equations (6) and (7) remain precisely the same, whether in terms of the dimensionless groups for a wall of height l used by the original authors, or using the groups of equations (3). Hence the wall solutions for heat transfer and fluid flow are represented on the channel diagrams by the same equations (6) and (7).

Elenbaas [5] in 1941 showed that a continuous solution exists which covers all conditions from the narrow channel through to the single wall, and numerical methods in the course of time have yielded such a solution, by Bodoia and Osterle [6] who in 1962 used a finite difference method. The basis is to obtain the velocity, temperature and pressure at assigned nodes, one row at a time, starting from the entrance and proceeding upward. The transverse velocity component is included by means of the continuity equation. At entry a uniform upward velocity and uniform temperature are prescribed. Pressure change along the channel is used to determine the channel height. Results are for $Pr = 0.70$, and Ra from about 0.1 to 10^4.

A finite element solution for laminar forced convection given by Tay and De Vahl Davis [10] in 1971 determined the temperature distribution in a channel for a constant property fluid in hydrodynamically developed flow, and entering at a uniform temperature. The solution employed the local potential as a variational criterion, with satisfactory results. The channel was made long enough for the exit to be in the thermally developed region, for which the temperature profile is known. This appears to place an undue restriction on the size of the channel.

In the present paper, progress towards a solution is made by way of two idealised cases. The first and more idealised case is termed "Stage 1", and uses two further simplifications. Fluid motion is allowed only in the upward (x) direction, and secondly the upthrust is taken to be due to a uniform fluid temperature within the channel. In the second case, termed "Stage 2", fluid motion is again only in the vertical direction, but upthrust is allowed to vary across the channel, and strictly compatible temperature and velocity variations obtained. The "full" problem with a second velocity component and a general variation of pressure, although much more complex, is within the scope of the methods described. This problem has been briefly attempted, but not yet with success. The objective of the present paper is to explore and test what may prove to be a more powerful analytical method than those previously used.

2. BASIC EQUATIONS COMMON TO STAGES 1 AND 2

A convective flow is induced through the channel by the action of a body force, in particular gravity. There is no internal heat generation. The Boussinesq approximation is employed, that is, viscous dissipation is neglected, fluid density is assumed constant except in forming the buoyancy term, and other fluid properties do not vary with temperature. Fluid motion is in the vertical (x) direction only. It follows that the velocity profile is the same from the channel entrance to the exit. Fluid temperatures increase with distance up the channel, with a region of heated fluid growing inward from each wall. Although steady-state conditions are being described, the governing equations are first of all written in the time-dependent form for derivation of the variational integrals.

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} - \frac{1}{\rho}\frac{\partial p}{\partial x} - g = 0 \tag{8}$$

$$\frac{\partial T}{\partial t} = \kappa\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) - u\frac{\partial T}{\partial x} = 0. \tag{9}$$

The conditions regarding velocity and density require that pressure varies only with x, and further that pressure gradient with x is uniform. If the depression required to produce the velocity profile is neglected, free stream pressure within the channel is identical with external hydrostatic pressure. In particular

$$\frac{\partial p}{\partial x} = -\rho_\infty g. \tag{10}$$

It is shown in Appendix 1 that for moderate temperature difference in a perfect gas, the error thus incurred in pressure fall is very small at low Gr, and increases to a small and fairly constant value, for example 0.04, for wide channel conditions. If desired, corrections may be applied during solution.

When the volume expansivity, β, is much less than 1, the fluid equation of state may be written in two forms:

$$\rho = \rho_\infty[1-\beta(T-T_\infty)] \tag{11}$$

$$\frac{\rho_\infty}{\rho} \doteqdot 1+\beta(T-T_\infty). \tag{12}$$

Now equations (10) and (12) may be used to substitute T for p in equation (8) to give

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2} + g\beta(T-T_\infty) = 0. \tag{13}$$

The boundary conditions at the channel walls are

$$\text{when } y = 0 \text{ and } y = b: \quad u = 0 \text{ and } T = T_w. \tag{14}$$

At $x = 0$ and $x = l$ no independent conditions are stipulated.

Introducing the dimensionless quantities

$$X = \frac{x}{l},\quad Y = \frac{y}{b},\quad \tau = \frac{t\nu Gr}{b^2},$$
$$B = \frac{b}{l},\quad U = \frac{ub^2}{\nu Grl},\quad \theta = \frac{T-T_\infty}{T_w - T_\infty} \tag{15}$$

the governing equations and boundary conditions become

$$Gr\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial Y^2} + \theta = 0 \tag{16}$$

$$Ra\frac{\partial \theta}{\partial \tau} = B^2\frac{\partial^2\theta}{\partial X^2} + \frac{\partial^2\theta}{\partial Y^2} - RaU\frac{\partial\theta}{\partial X} = 0 \tag{17}$$

$$\text{when } Y = 0 \text{ and } Y = 1,\quad U = 0 \text{ and } \theta = 1. \tag{18}$$

The characteristic form for natural convection, of coupled simultaneous equations in velocity and temperature, is retained.

3. ANALYSIS FOR STAGE 1

As a first approximation, it is assumed that all the fluid between the walls is at a uniform temperature equal to T_m, the mean value of an appropriate temperature distribution. This will be more nearly true for relatively narrow channels. Equation (16) reduces to

$$\frac{\partial^2 U}{\partial Y^2} + \theta_m = 0 \tag{19}$$

θ_m has a value between 0 and 1 which will be determined later. The resulting parabolic velocity profile holds for all values of Ra, Gr, Pr and B:

$$U = \frac{\theta_m}{2}(Y - Y^2). \tag{20}$$

Substitution into the energy equation (17) gives

$$Ra\frac{\partial\theta}{\partial\tau} = B^2\frac{\partial^2\theta}{\partial X^2} + \frac{\partial^2\theta}{\partial Y^2} - \frac{Ra}{2}\theta_m(Y-Y^2)\frac{\partial\theta}{\partial X} = 0 \tag{21}$$

$$\text{when } Y = 0 \text{ and } Y = 1,\quad \theta = 1. \tag{22}$$

These much more tractable uncoupled momentum and energy equations have the basic form of a forced convection problem.

Only an outline of the solution procedure is given in these paragraphs, and details are deferred to Appendix 2 of the paper. Following the Glansdorff and Prigogine [3] method of analysis, the equivalent functional E_{energy} is written for equation (21). This has the form of integrals in θ over the problem area and line integrals in θ along the boundaries. The distribution of θ which minimises E_{energy}, subject to the additional conditions imposed by equation (22), is the problem solution.

Boundary conditions at entry and exit are now considered. Fluid entry temperature has commonly been put equal to T_∞, which is not strictly correct. A more serious difficulty is to find a realistic independent condition for the exit. However the line integrals in E_{energy} constitute conditions at boundaries, which have a correspondence only to the governing equations, and are termed the "natural boundary conditions". Where there are no reliable independent boundary conditions, the natural boundary conditions can and should be used, and they together with the other terms of E_{energy} define the problem completely. This feature is justified as follows. Applying the Euler–Lagrange test

$$\frac{\partial E}{\partial \theta} - \frac{\partial}{\partial X}\left(\frac{\partial E}{\partial \theta_X}\right) - \frac{\partial}{\partial Y}\left(\frac{\partial E}{\partial \theta_Y}\right) = 0 \quad \text{where} \quad \theta_X = \frac{\partial \theta}{\partial X}, \quad \theta_Y = \frac{\partial \theta}{\partial Y} \tag{23}$$

for example to E_{energy} for Stage 1—see Appendix 2, equation (A6)—gives:

$$B^2 \frac{\partial^2 \theta}{\partial X^2} + \frac{\partial^2 \theta}{\partial Y^2} - \frac{Ra}{2}\theta_m (Y - Y^2)\frac{\partial \theta}{\partial X} = 0 \tag{24}$$

$$\frac{Ra}{2}\theta_m (Y - Y^2)\theta - B^2 \frac{\partial \theta}{\partial X} = 0, \quad \text{along } X = \text{constant} \tag{25}$$

$$\frac{\partial \theta}{\partial Y} = 0, \quad \text{along } Y = \text{constant}. \tag{26}$$

Equation (24) is the original energy equation (21) recovered, while equations (25) and (26) are the natural conditions at fluid boundaries parallel to the X and Y axes respectively, if no restriction is imposed at these boundaries. But when a specified condition is imposed at any boundary, the corresponding natural boundary condition no longer prevails—see the book by Schechter [2].

A finite element approximation is now applied. The problem area is divided into rectangular elements, and a simple variation of θ assigned within each element. Due to symmetry, only half the channel, from one wall to the centre plane, has to be considered. Figure 1 shows the division into 20 elements in the X-direction and 10 in the Y-direction used for Stage 1. The unknown quantities are now in general the values of θ at the 231 nodes selected. Along the mid-plane, nodes 11 to 231 are merely treated as internal nodes—see [1]. Along the wall at nodes 1 to 221, θ equals 1. At the entrance and exit, terms obtained from the natural boundary conditions are added. The resulting simultaneous equations in θ form a banded matrix with a relatively small band-width. The solution was obtained on an Elliott 4130 computer. The independent parameters are seen to be Rayleigh number and ratio B.

Finally the fluid mean temperature may be estimated. For each desired value of Rayleigh number and of width-to-height ratio B, an arbitrary uniform fluid temperature θ_u is given the values 0.1, 0.2, ..., 0.9, 1.0.

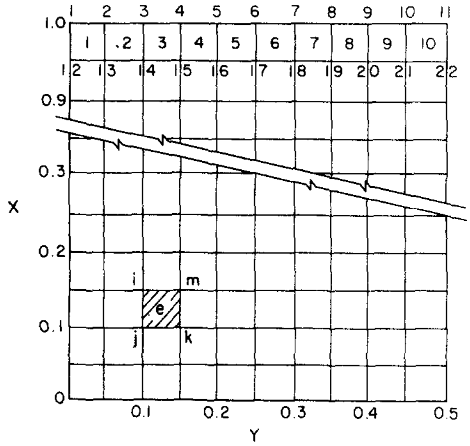

FIG. 1. Stage 1—division of half channel into 20 × 10 rectangular finite elements.

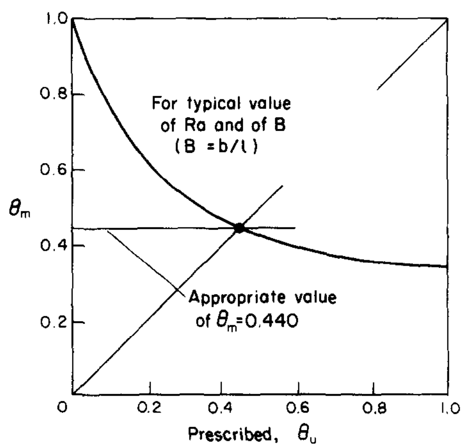

FIG. 2. Stage 1—determination of space mean fluid temperature.

The resulting temperature distributions are obtained, and space mean temperature θ_m set against θ_u as in Fig. 2. Equality of θ_m and θ_u indicates the best value of θ_m. The velocity profile and temperature distribution are then recalculated.

4. ANALYSIS FOR STAGE 2

In the pair of equations (16) and (17), with boundary conditions (18), both variables U and θ appear in both equations. A Stage 1 solution with any values of Ra, Gr and ratio B will serve as the starting point for a gross iteration between the two equations. This method was preferred to the use of a functional for both equations taken together. A less fine division into 10 × 10 elements (11 × 11 nodes) was used for Stage 2. The iteration procedure is as follows:

(a) In the parabolic velocity profile of a Stage 1 solution, the value $\theta_m = 0.5$ is taken in all cases for simplicity, that is

$$U = 0.25(Y - Y^2). \tag{27}$$

This velocity profile indicates an already available first approximate solution for θ. The choice of initial value for θ_m does not affect the final result.

(b) The mean temperature along each set of vertical meshes is used to obtain a superior approximate solution for U.

(c) The velocity at each column of nodes across the stream is used to obtain a superior approximate solution in θ.

(d) Reiteration of operations (b) and (c) is continued until both solutions converge sufficiently.

5. CORRELATION OF THEORETICAL RESULTS

The dimensionless rate of heat transfer from unit horizontal "depth" in the Z-direction from one wall to the rising fluid is

$$H = \int_0^{0.5} U\theta_1 \, dY \tag{28}$$

where θ_1 is the temperature profile at the channel exit ($X = 1$). Also the dimensionless volume flow rate of fluid from unit depth of the entire channel is

$$Q = 2\int_0^{0.5} U \, dY. \tag{29}$$

It is easily shown that

$$Nu = H \times Ra \tag{30}$$

and

$$Re = Q \times Gr. \tag{31}$$

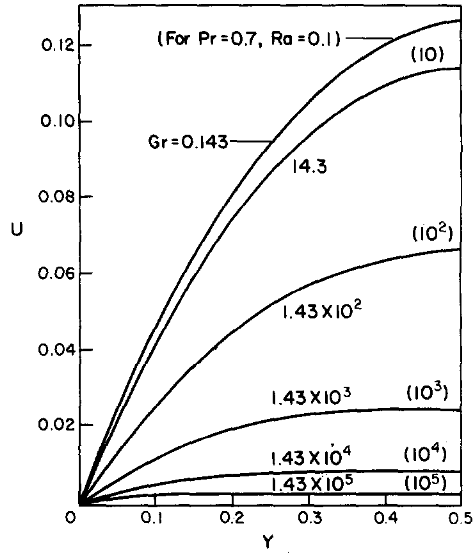

FIG. 3. Stage 2—fluid velocity. Range of Grashof number.

6. RESULTS

(1) Velocity and temperature solutions for both Stage 1 and Stage 2 were computed for values of width-to-height ratio B from 1:1 to 1:1000, and for Ra from 0.1 to 10^7, with $Pr = 0.7$ throughout, having gases in view. Variation of B was found to have a negligible

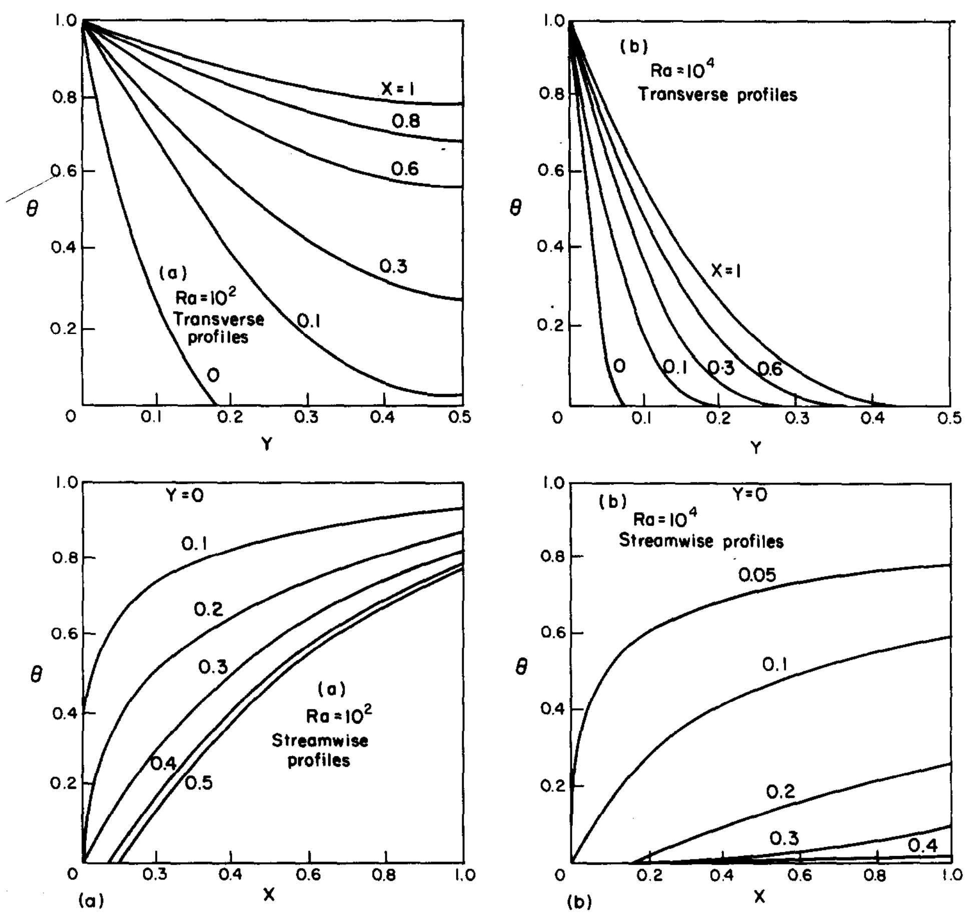

FIG. 4. Stage 2—fluid temperature. (a) $Ra = 10^2$; (b) $Ra = 10^4$.

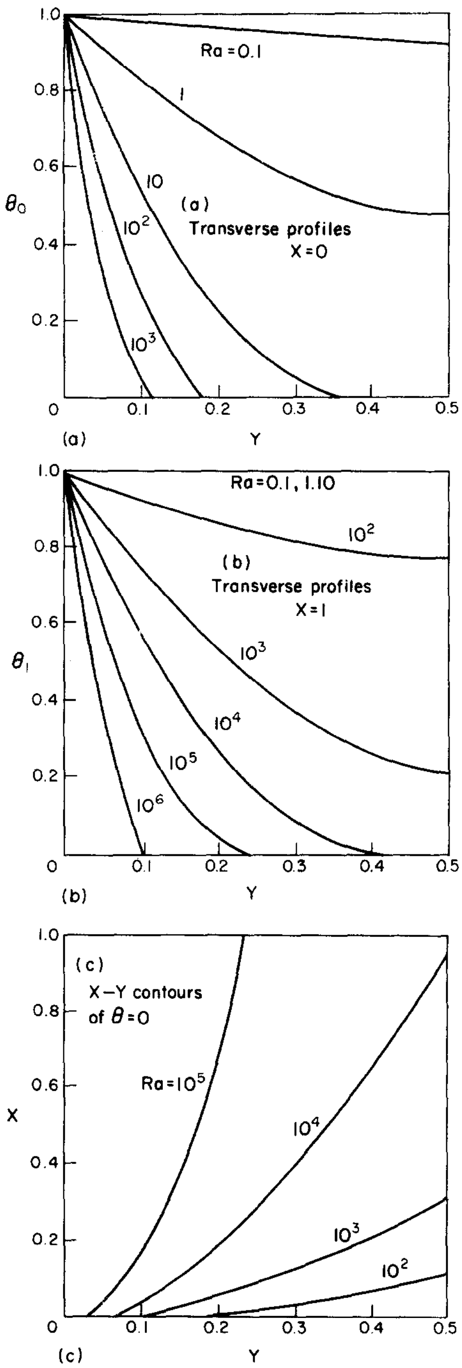

FIG. 5. Stage 2—fluid temperature. (a) Channel entrance: $Ra = 0.1$–10^3; (b) channel exit: $Ra = 0.1$–10^6; (c) extent of heated fluid: $Ra = 10^2$–10^5.

effect. Since B occurs only in the axial conduction term of equation (17), it follows that the contribution of axial conduction to the system heat transfer is negligible. Also the terms of the functionals used which contain B or B^2 may be omitted. However for the sake of generality, in Appendix 2 of this paper these terms are retained. Likewise, representative cases were computed both with and without the part of the natural boundary conditions representing axial conduction at entry and exit, and negligibly small effects found. These terms also may be discarded in this instance. Another very convenient property of the present system is that θ and Nu will give correlations against Ra, and U and Re will correlate against Gr, all independently of Pr. The values of U and θ for Stage 1 generally resemble those for Stage 2, and being of less interest, they are not presented in detail.

(2) Velocities and temperatures for Stage 2 are shown in Figs. 3–5. Figure 3 shows velocity profiles for most of the range of computation. For a gas with $Pr = 0.7$, the values of Gr taken give U corresponding to θ for $Ra = 0.1, 1 \ldots 10^5$. Stage 2 yields parabolic profiles only for Gr up to about 10^2, and thereafter the profiles become flatter. Figure 4 shows temperature distributions in the transition range between the narrow and the wide channel. The streamwise profiles particularly show the freedom given by use of the natural boundary conditions at entrance and exit. The temperatures at entrance show in Fig. 5(a) a varied and coherent solution instead of a flat zero profile. In Fig. 5(c) the region of heated fluid is shown extending below the channel entrance. Full temperature development at the exit, in so far as heated fluid reaches the mid-plane, is found for values of Ra up to 10^4.

(3) Overall heat transfer and fluid flow results: Fig. 6 shows Nu against Ra with logarithmic scales. For low Ra, all the channel relations merge with the narrow channel equation (4). For high Ra, the Stage 1 result remains surprisingly close to that for Stage 2. The latter is almost identical with the solution of Bodoia and Osterle [6] over the whole range of computation. Figure 7 shows Re against Gr with logarithmic scales. Again for low Gr, all the channel relations merge with the narrow channel equation (5). For high Gr, the Stage 1 result diverges markedly from that of Stage 2, which is somewhat higher than the Bodoia and Osterle solution. For high Ra and Gr respectively, the heat transfer and fluid flow results all have the form of simple power laws. These are collected in Table 1.

7. CONCLUSIONS

(1) The Stage 1 procedure was not expected to give realistic results for natural convection at high values of Ra and Gr. However it is very well suited for the analysis of laminar forced convection systems with hydrodynamically developed flow, since side flows are not present in these systems.

(2) The Stage 2 overall results for both heat transfer and fluid flow are quite satisfactory, despite the restriction that fluid flow may be vertical only. The agreement with the Bodoia and Osterle solution shows that the gross influence of this restriction is not large, in the conditions described.

(3) The status and capabilities of the local potential are perhaps not yet precisely known. However, the present results appear generally satisfactory, and also appear to account correctly for heat conduction and convection at boundaries within a fluid. It is a great advantage to have natural boundary conditions for use at the channel entry and exit.

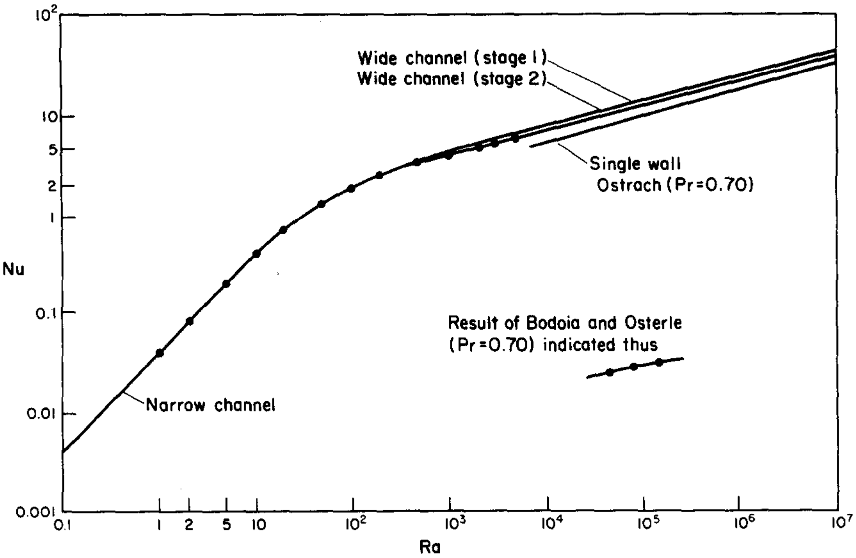

FIG. 6. Heat-transfer results: Nusselt number vs Rayleigh number.

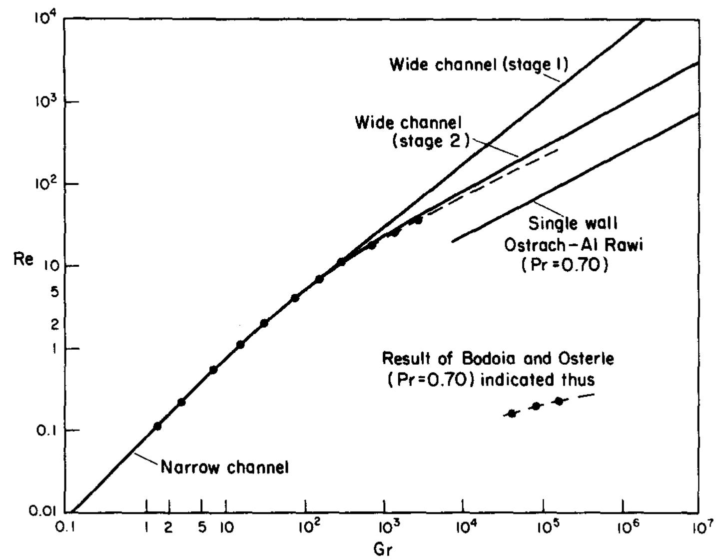

FIG. 7. Fluid flow results: Reynolds number vs Grashof number.

Table 1. Power law solutions for heat transfer at high Rayleigh number, and for fluid flow at high Grashof number

		Heat transfer	Fluid flow
Wide channel	Stage 1	$Nu = 0.750 Ra^{1/4}$	$^{*}Re = 0.160 Gr^{0.749}$
	Stage 2	$Nu = 0.699 Ra^{1/4}$	$Re = 0.725 Gr^{0.512}$
	Bodoia and Osterle [6] ($Pr = 0.7$)	$Nu = 0.680 Ra^{1/4}$	$Re = 0.817 Gr^{1/2}$
Single wall	Ostrach [8] and Al Rawi [9] ($Pr = 0.7$)	$Nu = 0.510 Ra^{1/4}$	$Re = 0.231 Gr^{1/2}$

*This relation is quite far from actuality.

(4) In the open vertical channel, at a moderate value of Gr a local velocity minimum appears at the mid-plane, and becomes more pronounced with increase in Gr. The Bodoia and Osterle method is capable of reproducing this feature, unlike the Stage 2 procedure, or any other which lacks flow normal to the walls. Recourse to the fuller system of equations is necessary for detailed representation of the wide channel.

(5) There are substantial differences between solutions for the wide channel and for the single wall. The wide channel values for heat transfer and fluid flow are greater by factors of 1.40 and 3.65 respectively. This disagreement will no doubt be resolved in due course.

Acknowledgements—The main parts of this work are taken from the Ph.D. thesis of Olusoji Ofi [11]. Gratitude is expressed to the University of Dundee for the award of a Research Studentship grant.

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APPENDIX 1

Pressure Fall in the Open Vertical Channel

Under the conditions of Stage 1 and Stage 2, the total fall in free stream pressure from entry to exit is

$$p_0 - p_l = \tfrac{1}{2}\rho u_{max}^2 + \rho g l. \tag{A1}$$

The proportional error, N, due to neglecting the first term will be evaluated. If N is small, it approximately equals the ratio of kinetic and hydrostatic pressure losses:

$$N = \frac{u_{max}^2}{u_{max}^2 + 2gl} \doteqdot \frac{u_{max}^2}{2gl} = \frac{1}{2}\left(\frac{u_{max}}{\bar{u}}\right)^2 \frac{Re^2}{Gr}\beta(T_w - T_\infty). \tag{A2}$$

We take a moderate temperature difference in a perfect gas, say $T_\infty = 300$, $T_w = 330$. Also $\beta = 1/T$.

$$\beta(T_w - T_\infty) \doteqdot 0.10. \tag{A3}$$

For narrow channels, $Re = \frac{1}{12}Gr$, $u_{max} = \frac{3}{2}\bar{u}$, $Gr \not> 10$.

$$N \doteqdot \tfrac{1}{128}Gr\beta(T_w - T_\infty) \not> \tfrac{1}{128}. \tag{A4}$$

For wide channels, $Re \doteqdot 0.9Gr^{1/2}$, and $u_{max} \doteqdot \bar{u}$

$$N \doteqdot 0.4\beta(T_w - T_\infty) \doteqdot 0.04. \tag{A5}$$

APPENDIX 2

Details of Variational and Finite Element Analysis

Stage 1

The equivalent functional for energy equation (21) is

$$E_{energy} = \iint_A \left[\tfrac{1}{2}B^2\left(\frac{\partial\theta}{\partial X}\right)^2 + \tfrac{1}{2}\left(\frac{\partial\theta}{\partial Y}\right)^2 - \frac{Ra}{2}\theta_m(Y-Y^2)\theta^0\frac{\partial\theta}{\partial X}\right]dX\,dY + \int_{cX}\left[\frac{Ra}{2}\theta_m(Y-Y^2)\theta^0\theta - B^2\frac{\partial\theta^0}{\partial X}\theta\right]dY - \int_{cY}\left[\frac{\partial\theta^0}{\partial Y}\theta\right]dX. \tag{A6}$$

θ^0 is defined as the value of θ which gives the stationary value to the functional E. During the minimization procedure, θ^0 is regarded as constant, but directly after taking the variation, we must put $\theta^0 = \theta$—see [2] and [3].

The finite elements are now formed. The fluid temperature within a typical element e is given as a bilinear function of X and Y:

$$\theta = A_1 + A_2X + A_3Y + A_4XY. \tag{A7}$$

By substituting the coordinates of the four nodes i, j, k, m of the element, we obtain a set of four simultaneous equations thus:

$$\begin{Bmatrix}\theta_i\\ \theta_j\\ \theta_k\\ \theta_m\end{Bmatrix} = \begin{bmatrix} 1, & X_i, & Y_i, & X_iY_i \\ & & \text{etc.} & \\ & & & \\ & & & \end{bmatrix}\begin{Bmatrix}A_1\\ A_2\\ A_3\\ A_4\end{Bmatrix} \tag{A8}$$

or simply $\{\theta\}^e = [T]\{A\}$ (A9)

giving $\{A\} = [T]^{-1}\{\theta\}^e$

and $\theta = [P]\{A\} = [P][T]^{-1}\{\theta\}^e$

where $[P] = [l, X, Y, XY]$

By writing $[N] = [P][T]^{-1}$

we have $\theta = [N]\{\theta\}^e = [N_i, N_j, N_k, N_m]\{\theta\}^e$ (A10)

where $[N]$ is a function only of nodal coordinates of the element.

The functional is now minimized. Substitution of the expression (A10) for θ into the expression (A6) for E, and differentiating E with respect to θ at node i of one element e gives

$$\frac{\partial}{\partial\theta_i}(E^e) = \iint_{A^e}\left[B^2\frac{\partial\theta}{\partial X}\frac{\partial}{\partial\theta_i}\left(\frac{\partial\theta}{\partial X}\right) + \frac{\partial\theta}{\partial Y}\frac{\partial}{\partial\theta_i}\left(\frac{\partial\theta}{\partial Y}\right) - \frac{Ra}{2}\theta_m(Y-Y^2)\theta\frac{\partial}{\partial\theta_i}\left(\frac{\partial\theta}{\partial X}\right)\right]dX\,dY + \frac{\partial}{\partial\theta_i}(E_s^e). \quad \text{(A11)}$$

Taking first of all the first term, which is the contribution of the internal nodes:

$$\frac{\partial}{\partial\theta_i}(E^e) = \iint_{A^e}\left[B^2\left(\frac{\partial N_i}{\partial X}\theta_i + \frac{\partial N_j}{\partial X}\theta_j + \dots\right)\frac{\partial N_i}{\partial X} + \left(\frac{\partial N_i}{\partial Y}\theta_i + \frac{\partial N_j}{\partial Y}\theta_j + \dots\right)\frac{\partial N_i}{\partial Y}\right.$$
$$\left. - \frac{Ra}{2}\theta_m(Y-Y^2)(N_i\theta_i + N_j\theta_j + \dots)\frac{\partial N_i}{\partial X}\right]dX\,dY \quad \text{(A12)}$$

or

$$\frac{\partial}{\partial\{\theta\}_i^e}(E^e) = [K]^e\{\theta\}^e \quad \text{(A13)}$$

where $[K]^e$ is a 4×4 element matrix such that

$$K_{ij} = \iint_{A^e}\left[B^2\frac{\partial N_i}{\partial X}\frac{\partial N_j}{\partial X} + \frac{\partial N_i}{\partial Y}\frac{\partial N_j}{\partial Y} - \frac{Ra}{2}\theta_m(Y-Y^2)N_j\frac{\partial N_i}{\partial X}\right]dX\,dY. \quad \text{(A14)}$$

The second term in expression (A11) is the contribution of nodes at the region boundaries. The part of the natural boundary conditions corresponding to heat conduction at the entrance and exit having been discarded, the only part remaining is that corresponding to heat convection. That is

$$\left.\begin{aligned}\frac{\partial}{\partial\theta_i}(E_s^e) &= \int_{cX}\frac{Ra}{2}\theta_m(Y-Y^2)\theta\frac{\partial\theta}{\partial\theta_i}dY\\ &= \int_{cX}\frac{Ra}{2}\theta_m(Y-Y^2)(N_i\theta_i + N_j\theta_j + \dots)N_i\,dY.\end{aligned}\right\} \quad \text{(A15)}$$

Adding the contributions of all the elements surrounding node i and setting to zero gives the minimization equation in θ at that node. When there are n nodes, the system of linear equations thus obtained is of the form

$$[B]\begin{Bmatrix}\theta_1\\ \theta_2\\ \vdots\\ \theta_n\end{Bmatrix} = \{C\} \quad \text{(A16)}$$

and can be solved for values of θ on the computer.

Introduction of the imposed boundary condition completes the finite element formulation. Along the wall surface where the values of θ are specified as unity, the terms of the corresponding rows and columns in the system matrix are all set to zero except the diagonal terms which are set to unity, as are also the corresponding terms in the column matrix C.

The matrix of equation (A16) is banded. That is, the non-zero terms are concentrated around the leading diagonal of the matrix. This is a consequence of the manner in which it has been assembled from the contributions of each element. This effect is used to save computer time by storing only the rectangular band. Solution was by a modified Gaussian elimination method with back substitution.

Stage 2

In the momentum equation (16), θ is a known function of Y. The equivalent functional is

$$E_{\text{mom}} = \int_0^{0.5}\left[\frac{1}{2}\left(\frac{\partial U}{\partial Y}\right)^2 - \theta^0 U\right]dY. \quad \text{(A17)}$$

There is also the imposed boundary condition that

$$U = 0 \quad \text{when} \quad Y = 0. \quad \text{(A18)}$$

A solution is determined for U at 11 nodes evenly spaced from $Y = 0$ to $Y = 0.5$. These nodes form the bounds of 10 one-dimensional finite elements. When E is written for one element and the variation of U taken, the contribution at one node i of the element is

$$\frac{\partial}{\partial U_i}(E_{\text{mom}}^e) = \int_0^{0.5}\left[\left(\frac{\partial N_i}{\partial Y}U_i + \frac{\partial N_j}{\partial Y}U_j + \dots\right)\frac{\partial N_i}{\partial Y} - \theta N_i\right]dY \quad \text{(A19)}$$

or

$$\frac{\partial}{\partial\{U\}_i^e}(E^e_{\mathrm{mom}}) = [K_i] \quad \{U\} - \{h\}. \tag{A20}$$

In the energy equation (17), U is a known function of Y, and the equivalent functional is

$$E_{\mathrm{energy}} = \iint_A \left[\frac{1}{2}B^2\left(\frac{\partial\theta}{\partial X}\right)^2 + \frac{1}{2}\left(\frac{\partial\theta}{\partial Y}\right)^2 - Ra\,U^0\theta^0\frac{\partial\theta}{\partial X}\right]\mathrm{d}X\,\mathrm{d}Y + \int_{cX}\left[Ra\,U^0\theta^0\theta - B^2\frac{\partial\theta}{\partial X}\theta\right]\mathrm{d}Y - \int_{cY}\frac{\partial\theta^0}{\partial Y}\theta\,\mathrm{d}X. \tag{A21}$$

A solution is determined for θ at the 121 nodes at the corners of a 10×10 array of rectangular finite elements in the X–Y plane. After taking the variation of θ, the contribution from each element for one node i is

$$\frac{\partial}{\partial\theta_i}(E^e) = \iint_{A^e}\left[B^2\left(\frac{\partial N_i}{\partial X}\theta_i + \frac{\partial N_j}{\partial X}\theta_j + \ldots\right)\frac{\partial N_i}{\partial X} + \left(\frac{\partial N_i}{\partial Y}\theta_i + \frac{\partial N_j}{\partial Y}\theta_j + \ldots\right)\frac{\partial N_i}{\partial Y} - RaU(N_i\theta_i + N_j\theta_j + \ldots)\frac{\partial N_i}{\partial X}\right]\mathrm{d}X\,\mathrm{d}Y + \frac{\partial}{\partial\theta_i}(E^e_s). \tag{A22}$$

The remaining parts of the two separate solution procedures are as already described for Stage 1.

APPLICATION DE LA METHODE DES ELEMENTS FINIS A LA CONVECTION THERMIQUE NATURELLE DANS LES CANAUX VERTICAUX ET OUVERTS

Résumé—On utilise la méthode des éléments finis en même temps que le principe variationel du potentiel local pour obtenir des solutions numériques approchées pour la convection laminaire stationnaire dans un canal vertical et ouvert, avec une température pariétale uniforme. Les conditions aux limites naturelles générées par la technique variationnelle sont données à l'entrée et à la sortie du canal. On a observé la restriction d'une vitesse du fluide verticale.

Les températures et les vitesses ascentionnelles sont obtenues pour des nombres de Rayleigh et de Grashof variant entre 0,1 et 10^7. Les corrélations entre les nombres de Nusselt et de Reynolds sont en bon accord avec la solution aux différences finies de Bodoia et Osterle, et en moins bon accord avec une solution de paroi unique verticale.

ANWENDUNG DER METHODE DER FINITEN ELEMENTE AUF DEN WÄRMEÜBERGANG BEI NATÜRLICHER KONVEKTION AN OFFENEN, VERTIKALEN KANÄLEN

Zusammenfassung—Zur näherungsweisen numerischen Lösung der stationären, natürlichen Konvektion in offenen vertikalen Kanälen mit gleichförmiger Wandtemperatur wird die Methode der finiten Elemente zusammen mit dem Variationsprinzip für das lokale Potential verwendet. Die durch die Variationstechnik bedingten natürlichen Randbedingungen werden für Kanalein- und -austritt formuliert; die Untersuchung bleibt auf vertikale Strömungsgeschwindigkeiten beschränkt. Die Temperaturen und die aufwärtsgerichteten Strömungsgeschwindigkeiten werden für Rayleigh- und Grashof-Zahlen von 0,1 bis 10^7 ermittelt. Die Nusselt–Reynolds-Beziehungen stimmen gut mit der mit finiten Differenzen gewonnenen Lösung von Bodoia und Osterle überein; stärkere Abweichungen ergeben sich im Vergleich mit der Lösung für die einzelne, vertikale Wand.

ПРИМЕНЕНИЕ МЕТОДА КОНЕЧНЫХ ЭЛЕМЕНТОВ ДЛЯ РЕШЕНИЯ ЗАДАЧ КОНВЕКТИВНОГО ТЕПЛООБМЕНА В ОТКРЫТОМ ВЕРТИКАЛЬНОМ КАНАЛЕ

Аннотация — Метод конечных элементов вместе с вариационным принципом локального потенциала используется для получения приближенных численных решений задач стационарной ламинарной естественной конвекции в открытом вертикальном канале с постоянной температурой на стенках. Граничные условия на входе и выходе из канала получаются вариационными методами. Предполагается, что скорость течения имеет только вертикальную компоненту. Получены температура и подъемная скорость для диапазона чисел Рэлея и Грасгофа от 0,1 до 10^7. Зависимости чисел Нуссельта и Рейнольдса хорошо совпадают с аналогичными зависимостями, полученными Бодоя и Остерле, и хуже с решением, полученным для вертикальной стенки.